Brudno's theorem for $\mathbb{Z}^d$ (or $\mathbb{Z}^d_+$) subshifts

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Abstract

We generalize Brudno's theorem of 1-dimensional shift dynamical system to $\mathbb{Z}^d$ (or $\mathbb{Z}^d_+$) subshifts. That is to say, in $\mathbb{Z}^d$ (or $\mathbb{Z}^d_+$) subshift, the Kolmogorov-Sinai entropy is equivalent to the Kolmogorov complexity density almost everywhere for an ergodic shift-invariant measure.

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1 Introduction

In a topological dynamical system, A. A. Brudno defined a complexity of the trajectory of a point in the space by using the notion of Kolmogorov complexity, and showed the relationship between this quantity and the Kolmogorov-Sinai entropy [2]. As a preliminary step, Brudno considered the 1-dimensional shift dynamical system and showed that, for an ergodic shift-invariant measure, the Kolmogorov complexity density is equal to the Kolmogorov-Sinai entropy almost everywhere [2, Theorem 1.1].

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A partial approach to generalize this theorem to a d-dimensional case is found in [7]. S. G. Simpson showed that, in $\mathbb{Z}^d$ (or $\mathbb{Z}_+^d$) subshifts, there exists a point such that its Kolmogorov complexity density is coincident with the topological entropy [7]. Examining Simpson's proof, we see that what he showed substantively is that the Kolmogorov complexity density is equal to the Kolmogorov-Sinai entropy almost everywhere only for a measure of maximal entropy.

The purpose of this paper is to generalize the Brudno's theorem of the $\mathbb{Z}_+^1$-action shift dynamical system to $\mathbb{Z}^d$ (or $\mathbb{Z}_+^d$) subshifts. The main theorem is the following:

Theorem 3.1 (Brudno's theorem for $\mathbb{Z}^d$ (or $\mathbb{Z}_+^d$) subshifts) *If $\mu \in EM(S, \varsigma)$, then*

$$\mathcal{K}(\omega) = h_\varsigma(\mu), \quad \mu\text{-}a.e.\omega \in S. \tag{3.1}$$

Here S denotes $\mathbb{Z}^d$ (or $\mathbb{Z}_+^d$) subshift, ς denotes the shift action on S, $EM(S, \varsigma)$ denotes the set of all ergodic shift-invariant measures on the topological dynamical system (S, ς), $\mathcal{K}(\omega)$ denotes the Kolmogorov complexity density of ω, and $h_\varsigma(\mu)$ denotes the Kolmogorov-Sinai entropy of the measure preserving dynamical system $(S, \mathfrak{B}(S), \mu, \varsigma)$. We give the rigorous definition of these terms in Section 2.

In Section 2, we introduce some basic mathematical notions in ergodic theory, Kolmogorov complexity and shift dynamical systems. We used [3] and [4] as main references for this section. Using these basic notions, we define the Kolmogorov complexity density of each point of $\Sigma^{\mathbb{Z}^d}$ (or $\Sigma^{\mathbb{Z}_+^d}$) naturally.

In Section 3, we prove the main theorem. The proof directly uses an idea of Brudno's original paper, i.e., Shannon-McMillan-Breimann theorem and the notion of frequency set.

In the last section, as an application of the main theorem, we show a variational principle using the Kolmogorov complexity density.

2 Some Mathematical Preliminaries

We first give quick reviews for some mathematical results related to the main theorem. Descriptions of this section are restricted to a minimum and all the contents in this section are well known. We write

$$\mathbb{N} = \{1, 2, \cdots\}, \ \mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}, \ \mathbb{Z}_+ = \{0, 1, 2, \cdots\}.$$

For an arbitrary fixed $d \in \mathbb{N}$, we set $G := \mathbb{Z}^d$ or $G := \mathbb{Z}_+^d$. For all $n \in \mathbb{N}$, let

$$\Lambda_n := \{g = (g_i)_{i=1}^d \in G : \forall i \in \{1, \cdots, d\}, |g_i| < n\}.$$

Then we have

$$|\Lambda_n| = \begin{cases} (2n-1)^d & (G = \mathbb{Z}^d), \\ n^d & (G = \mathbb{Z}_+^d), \end{cases}$$

where we denote by $|A|$ the cardinality of a set A.

2.1 Ergodic theory

2.1.1 Measure preserving dynamical system

First, let us define Kolmogorov-Sinai entropy, also known as measure-theoretic entropy.

Definition 2.1 (Measure preserving dynamical system) *Let $(X, \mathfrak{B}, \mu)$ be a probability space and $\mathcal{T} = (T^g)_{g \in G}$ be a family of maps on X such that*

(1) μ is $\mathcal{T}$-invariant, i.e., $\forall g \in G, \forall A \in \mathfrak{B}, \mu(T^{-g}A) = \mu(A)$ (where $T^{-g} := (T^g)^{-1}$);

(2) $\mathcal{T}$ is a measurable action of G on X, i.e., $T^g : X \to X$ is measurable for all $g \in G$, $T^0 = I_X$ (the identity map on X) and $\forall g, g' \in G, T^{g+g'} = T^g \circ T^{g'}$ (if $G = \mathbb{Z}^d$, then $(T^g)^{-1} = T^{-g}$ holds).

We call such a quadruple $(X, \mathfrak{B}, \mu, \mathcal{T})$ a measure preserving dynamical system (m.p.d.s.).

Definition 2.2 *Let $(X, \mathfrak{B}, \mu, \mathcal{T})$ be a m.p.d.s.. A $\mathfrak{B}$-measurable function f on X is said to be $\mathcal{T}$-invariant $\bmod \mu$ if and only if $\forall g \in G, f \circ T^g = f$ (μ-a.s.). A set $A \in \mathfrak{B}$ is said to be $\mathcal{T}$-invariant $\bmod \mu$ if and only if 1_A is $\mathcal{T}$-invariant $\bmod \mu$, where we denote the characteristic function of A by 1_A. We write $\mathcal{I}_\mu(\mathcal{T}) := \{A \in \mathfrak{B} : A \text{ is } \mathcal{T}\text{-invariant} \bmod \mu\} = \{A \in \mathfrak{B} : \forall g \in G, \mu(T^{-g}A \triangle A) = 0\}$, where $\triangle$ denotes the symmetric difference.*

Theorem 2.3 (Birkhoff's ergodic theorem) *Let $(X, \mathfrak{B}, \mu, \mathcal{T})$ be a m.p.d.s.. Then, for all $f \in L^1(X, \mu)$, there exists the limit*

$$\bar{f}(x) := \lim_{n \to \infty} \frac{1}{|\Lambda_n|} \sum_{g \in \Lambda_n} f(T^g x), \quad \mu\text{-a.s. } x,$$

and $\bar{f} \in L^1(X,\mu)$*. Moreover,* $\bar{f}$ *is* $\mathfrak{T}$*-invariant* $\bmod \mu$ *and*

$$\forall A \in \mathfrak{I}_\mu(\mathfrak{T}), \ \int_A \bar{f} d\mu = \int_A f d\mu.$$

Proof. See [3]. □

Definition 2.4 (Ergodicity) *Let* $(X, \mathfrak{B}, \mu, \mathfrak{T})$ *be a m.p.d.s. If for all* $f \in L^1(X,\mu)$

$$\bar{f} = \int_X f d\mu, \quad \mu\text{-a.s. } x$$

holds, then the m.p.d.s. $(X, \mathfrak{B}, \mu, \mathfrak{T})$ *is said to be ergodic. In this case,* μ *is called an ergodic* $\mathfrak{T}$*-invariant probability measure on the measurable space* $(X, \mathfrak{B})$*.*

Although there are several equivalent conditions of ergodicity, only the above-mentioned condition is used in this paper.

Definition 2.5 (μ-partition) *Let* $(X, \mathfrak{B}, \mu)$ *be a probability space. A family of measurable sets* $\alpha = \{A_i : i \in I\} \subset \mathfrak{B}$ *is called a* μ*-partition of* X *if the following conditions hold:*

$$\mu(A_i \cap A_j) = 0 \ (i \neq j), \ \mu\left(X \setminus \bigcup_{i \in I} A_i\right) = 0 \text{ and } \mu(A_i) > 0 \ (\forall i \in I).$$

Accordingly, α *is at most countable. If* $|I| < \infty$ *is holds, then* α *is called a finite* μ*-partition.*

Let α and β be μ-partitions of X. The common refinement of α and β

$$\alpha \vee \beta := \{A \cap B : A \in \alpha, B \in \beta, \mu(A \cap B) > 0\}$$

is a μ-partition of X.

Definition 2.6 (Information and entropy of a μ-partion) *Let* $(X, \mathfrak{B}, \mu)$ *be a probability space, and* α *be a* μ*-partition of* X*. The information of* α *is the measurable function* I_α *on* X *defined by*

$$I_\alpha(x) := -\sum_{A \in \alpha} \log_2 \mu(A) \cdot 1_A(x), \quad x \in X.$$

The entropy of α is defined by the average information, i.e.,

$$H(\alpha) := \int_X I(\alpha) d\mu = \sum_{A \in \alpha} \varphi(\mu(A)),$$

where we define the function $\varphi : [0, \infty) \to \mathbb{R}$ by

$$\varphi(t) := \begin{cases} -t \log_2 t & (t > 0), \\ 0 & (t = 0). \end{cases}$$

From Kolmogorov complexity's point of view, we choose the binary logarithm $\log_2$ instead of $\log_e$.

Definition 2.7 (Dynamical entropy relative to a partition) *Let $(X, \mathfrak{B}, \mu, \mathfrak{T})$ be a m.p.d.s. and α be a μ-partition of X. We set $T^{-g}\alpha := \{T^{-g}A : A \in \alpha\}$ for each $g \in G$ and $\alpha^{\Lambda} := \bigvee_{g \in \Lambda} T^{-g}\alpha$ for a finite subset $\Lambda \subset G$. The dynamical entropy of the m.p.d.s. $(X, \mathfrak{B}, \mu, \mathfrak{T})$ relative to the partition α is defined by*

$$h(\mu, \alpha, \mathfrak{T}) := \inf_{n>0} \frac{1}{|\Lambda_n|} H(\alpha^{\Lambda_n}).$$

Theorem 2.8 *Let $(X, \mathfrak{B}, \mu, \mathfrak{T})$ be a m.p.d.s. and α be a μ-partition of X. Then*

$$h(\mu, \alpha, \mathfrak{T}) = \lim_{n \to \infty} \frac{1}{|\Lambda_n|} H(\alpha^{\Lambda_n}).$$

Proof. See [3]. □

Theorem 2.9 (Shannon-McMillan-Breiman) *Let $(X, \mathfrak{B}, \mu, \mathfrak{T})$ be an ergodic m.p.d.s. and α be a μ-partition of X with $H(\alpha) < \infty$. Then*

$$h(\mu, \alpha, \mathfrak{T}) = \lim_{n \to \infty} \frac{1}{|\Lambda_n|} I_{\alpha^{\Lambda_n}} \quad \text{in } L^1(X, \mu).$$

Moreover, if α is finite, then this convergence holds also for μ-a.s.

Proof. See [3, 5]. □

Definition 2.10 (Kolmogorov-Sinai entropy) *The Kolmogorov-Sinai entropy (KS entropy) of the m.p.d.s $(X, \mathfrak{B}, \mu, \mathfrak{T})$ is defined by*

$$h_{\mathfrak{T}}(\mu) := \sup\{h(\mu, \alpha, \mathfrak{T}) : \alpha \text{ is a } \mu\text{-partition with } H(\alpha) < \infty\}.$$

Definition 2.11 (μ-generator) *Let $(X, \mathfrak{B}, \mu, \mathfrak{T})$ be a m.p.d.s.. A μ-partition α is called a μ-generator if $\alpha^G = \mathfrak{B} \mod \mu$, where this equation means that $\forall A \in \mathfrak{B}, \exists B \in \alpha^G, \mu(A \triangle B) = 0$.*

Theorem 2.12 (Kolmogorov-Sinai) *Let $(X, \mathfrak{B}, \mu, \mathfrak{T})$ be a m.p.d.s. and α be a μ-generator such that $H(\alpha) < \infty$. Then $h_{\mathfrak{T}}(\mu) = h(\mu, \alpha, \mathfrak{T})$.*

Proof. See [3]. □

2.1.2 Topological dynamical system

We give the definition of topological dynamical system and its entropy via a variational principle of KS entropy.

Definition 2.13 (Topological dynamical system) *The pair $(X, \mathfrak{T})$ is called a topological dynamical system (t.d.s.) if the following conditions hold:*

(1) X is a compact metrizable space;

(2) $\mathfrak{T} = (T^g)_{g \in G}$ is a continuous action of G on X, i.e., for all $g \in G$, $T^g : X \to X$ is continuous.

In this setting we denote by $\mathfrak{B}(X)$ the Borel σ-algebra of X. By (2), $\mathfrak{T}$ is a measurable action of G on X.

Let $(X, \mathfrak{T})$ be a t.d.s.. We denote by $M(X)$ the set of all probability measures on the Borel measurable space $(X, \mathfrak{B}(X))$, by $M(X, \mathfrak{T})$ the set of all $\mathfrak{T}$-invariant probability measures on $(X, \mathfrak{B}(X))$ and by $EM(X, \mathfrak{T})$ the set of all ergodic members in $M(X, \mathfrak{T})$, respectively.

Theorem 2.14 (Krylov-Bogolubov) *If $X \neq \emptyset$ and $(X, \mathfrak{T})$ is a t.d.s., then $M(X, \mathfrak{T}) \neq \emptyset$.*

Proof. See [3]. □

Obviously, if $\mu \in M(X, \mathfrak{T})$ then $(X, \mathfrak{B}(X), \mu, \mathfrak{T})$ is a m.p.d.s.. We define a concept of topological entropy of a t.d.s. $(X, \mathfrak{T})$ as follows.

Definition 2.15 (upper semicontinuous function) *Let Y be a topological space. We set*

$$USC(Y) := \{f : Y \to [-\infty, \infty) : \forall c \in \mathbb{R}, \{y \in Y : f(y) < c\} \text{ is open}\},$$

and an element of $USC(Y)$ is called an upper semicontinuous function on Y.

Definition 2.16 (Pressure, topological entropy, equilibrium state) *Let $(X, \mathfrak{T})$ be a t.d.s. and let $\psi \in USC(X)$, $\inf \psi > -\infty$. The pressure of ψ is defined by*

$$p(\psi) := \sup_{\mu \in M(X,\mathfrak{T})} (h_{\mathfrak{T}}(\mu) + \mu(\psi))$$

where $\mu(\psi) := \int_X \psi(x) d\mu(x)$. A measure $\nu \in M(X, \mathfrak{T})$ is called an equilibrium state for $\psi \in USC(X)$ if

$$p(\psi) = h_{\mathfrak{T}}(\nu) + \nu(\psi).$$

In particular, $p(0) = \sup_{\mu \in M(X,\mathfrak{T})} h_{\mathfrak{T}}(\mu)$ is called the topological entropy of $(X, \mathfrak{T})$, and the equilibrium state for $\psi = 0$ is called a measure of maximal entropy for $\mathfrak{T}$.

Theorem 2.17 (Ergodic decomposition) *Let $(X, \mathfrak{T})$ be a t.d.s.. Then, for each $\mu \in M(X, \mathfrak{T})$, there uniquely exists a measure ρ on the space $M(X, \mathfrak{T})$ (with respect to the Borel σ-algebra associated to the weak-$*$ topology) such that*

(1) for any bounded measurable function $f : X \to \mathbb{R}$ we have

$$\int_X f(x) d\mu(x) = \int_{EM(X,\mathfrak{T})} \left\{ \int_X f(x) d\nu(x) \right\} d\rho(\nu).$$

(2) $\rho(EM(X, \mathfrak{T})) = 1$.

Proof. See [3, 6, 8]. □

Since $\mu(A) = \int_{EM(X,\mathfrak{T})} \nu(A) d\rho(\nu)$ for a measurable set $A \in \mathfrak{B}(X)$, we write $\mu = \int_{EM(X,\mathfrak{T})} \nu d\rho(\nu)$ which is called the ergodic decomposition of μ.

Theorem 2.18 (Jacobs's theorem) *Let $(X, \mathfrak{T})$ be a t.d.s.. If $\mu \in M(X, \mathfrak{T})$ and $\mu = \int_{EM(X,\mathfrak{T})} \nu d\rho(\nu)$ is the ergodic decomposition of μ, then we have*

$$h_{\mathfrak{T}}(\mu) = \int_{EM(X,\mathfrak{T})} h_{\mathfrak{T}}(\nu) d\rho(\nu).$$

Proof. See [3, 8]. □

2.2 Kolmogorov complexity

Let $\mathcal{A}$ be a nonempty finite set. Without loss of generality, we set $\mathcal{A} := \{0, 1, \cdots, N\}$ where $N \in \mathbb{Z}_+$.

We define the set of all finite *strings* over $\mathcal{A}$ as follows.

$$\mathcal{A}^* := \bigcup_{n=0}^{\infty} \mathcal{A}^n = \{\lambda, 0, 1, \cdots, N, 00, 01, \cdots, 0N, 10, \cdots, 1N, \cdots, NN, 000, \cdots\}$$

where $\mathcal{A}^0 = \{\lambda\}$ and λ denote the empty string.

We often identify $\mathcal{A}^*$ with $\mathbb{Z}_+$ or $\mathbb{Z}$ by using the bijective map $I_{\mathcal{A}^* \to \sharp} : \mathcal{A}^* \to \sharp$ ($\sharp \in \{\mathbb{Z}_+, \mathbb{Z}\}$) defined by the following.

$$\begin{aligned} I_{\mathcal{A}^* \to \mathbb{Z}_+}(x) &:= \begin{cases} \sum_{k=0}^{n-1} (N+1)^k + \sum_{k=1}^{n} a_k (N+1)^{n-k}, & x = a_1 a_2 \cdots a_n \in \mathcal{A}^n \ (n \in \mathbb{N}), \\ 0, & x = \lambda, \end{cases} \\ I_{\mathcal{A}^* \to \mathbb{Z}}(x) &:= \alpha\left(I_{\mathcal{A}^* \to \mathbb{Z}_+}(x)\right) \end{aligned}$$

where $\alpha(n) := (-1)^{n+1} \lfloor \frac{n+1}{2} \rfloor$ for all $n \in \mathbb{Z}_+$.

For example, the case of $\mathcal{A} = \{0, 1\}$ is as follows:

x	λ	0	1	00	01	10	11	000	001	$\cdots$
$I_{\{0,1\}^* \to \mathbb{Z}_+}(x)$	0	1	2	3	4	5	6	7	8	$\cdots$
x	λ	0	1	00	01	10	11	000	001	$\cdots$
$I_{\{0,1\}^* \to \mathbb{Z}}(x)$	0	1	-1	2	-2	3	-3	4	-4	$\cdots$

For convenience, we define $I_{\sharp \to \mathcal{A}^*} := I^{-1}_{\mathcal{A}^* \to \sharp}$.

The map $\mathcal{A}^* \times \mathcal{A}^* \ni (x, y) \mapsto xy \in \mathcal{A}^*$ is called the *concatenation*. The set $\mathcal{A}^*$ with the concatenation is a monoid with identity element λ, i.e., $(xy)z = x(yz)$ for all $x, y, z \in \mathcal{A}^*$ and $\lambda x = x\lambda = x$ for all $x \in \mathcal{A}^*$.

The *length* of $x \in \mathcal{A}^*$ is denoted by $l(x)$ which is defined by $l(x) = n \overset{\text{def}}{\Leftrightarrow} x \in \mathcal{A}^n$. Obviously, we have for all $x, y \in \mathcal{A}^*, l(xy) = l(x) + l(y)$.

For all $x, y \in \mathcal{A}^*$, we call x a *prefix* of y if there exists $z \in \mathcal{A}^*$ such that $y = xz$. A set $A \subset \mathcal{A}^*$ is said to be *prefix-free* if, for all $x \in A$, the elements of $A \setminus \{x\}$ is not a prefix of x. We set for all $x \in \mathcal{A}^*$

$$\bar{x} := \underbrace{1 \cdots 1}_{l(x)} 0x$$

then we have $l(\bar{x}) = 2l(x) + 1$.

Let $\mathcal{A}_1, \mathcal{A}_2$ be a nonempty finite set. Let $\mathcal{D}$ be a subset of $\mathcal{A}_1^*$ and let f be a function from $\mathcal{D}$ to $\mathcal{A}_2^*$. If $\mathcal{D} \subsetneq \mathcal{A}_1^*$, we call such a function f a *partial function* and write $f : \mathcal{A}_1^* \rightsquigarrow \mathcal{A}_2^*$, and if $\mathcal{D} = \mathcal{A}_1^*$ then we call f a *total function.*

A partial function $\phi : \mathcal{A}^* \rightsquigarrow \mathcal{A}^*$ is said to be *partial recursive* if and only if there exists a Turing machine M such that ϕ is computed by M, i.e., for all $x \in \mathcal{A}^*$, M halts if and only if $x \in \mathrm{dom}(\phi)$, in that case, M outputs $\phi(x)$.

A partial function $\phi : \mathcal{A}_1^* \rightsquigarrow \mathcal{A}_2^*$ is partial recursive if there exists a partial recursive function $\psi : \mathcal{A}_1^* \rightsquigarrow \mathcal{A}_1^*$ such that $\phi = I_{\mathbb{Z}_+ \to \mathcal{A}_2^*} \circ I_{\mathcal{A}_1^* \to \mathbb{Z}_+} \circ \psi$.

A *partial recursive prefix function* $\phi : \mathcal{A}_1^* \rightsquigarrow \mathcal{A}_2^*$ is a partial recursive function such that $\mathrm{dom}(\phi)$ is prefix-free.

Let $\phi : \{0,1\}^* \rightsquigarrow \mathcal{A}^*$ be a partial recursive prefix function. For all $x \in \mathcal{A}^*$, the *complexity* of x with respect to ϕ is defined by

$$K_\phi(x) := \begin{cases} \min\{l(p) : p \in \phi^{-1}(x)\}, & (\phi^{-1}(x) \neq \emptyset), \\ \infty & (\phi^{-1}(x) = \emptyset). \end{cases}$$

A partial recursive prefix function $\phi : \{0,1\}^* \rightsquigarrow \mathcal{A}^*$ is said to be *additively optimal* if for all partial recursive prefix function $\psi : \{0,1\}^* \rightsquigarrow \mathcal{A}^*$, there exists a constant $c_{\phi,\psi} \in \mathbb{R}$ such that

$$\forall x \in \mathcal{A}^*, \quad K_\phi(x) \leq K_\psi(x) + c_{\phi,\psi}.$$

Theorem 2.19 *There exists an additively optimal partial recursive prefix function.*

Proof. See [4]. □

For each pair (ϕ, ψ) of additively optimal partial recursive prefix functions from $\{0,1\}^*$ to $\mathcal{A}^*$, there exists a constant $c_{\phi,\psi} > 0$ such that for all $x \in \mathcal{A}^*$, $|K_\phi(x) - K_\psi(x)| \leq c_{\phi,\psi}$.

It is easily seen that any additively optimal partial recursive prefix function is surjective.

Definition 2.20 *We fix one additively optimal partial recursive prefix function $\phi : \{0,1\}^* \rightsquigarrow \mathcal{A}^*$. We define the prefix Kolmogorov complexity of $x \in \mathcal{A}^*$ by*

$$K(x) := K_\phi(x).$$

2.3 Shift dynamical system

Let Σ be a nonempty finite set, and we set $\Omega := \Sigma^G$. By Tychonoff's theorem, Ω endowed with the product topology of the discrete topology on Σ is a compact topological space. It is well-known that this topology is also generated by the metric $d(\omega, \omega') = 2^{-n(\omega,\omega')}$ where $n(\omega, \omega') = \sup\{n \in \mathbb{N} : \forall g \in \Lambda_n, \omega_g = \omega'_g\}$ for all $\omega = (\omega_g)_{g\in G}, \omega' = (\omega'_g)_{g\in G} \in \Omega$. For all $n \in \mathbb{N}$ and for all $s \in \Sigma^{\Lambda_n}$, we define the cylinder set of s by $[\![s]\!] := \{\omega \in \Omega : \omega \restriction \Lambda_n = s\}$ where $\omega \restriction \Lambda_n$ denotes the restriction of ω to Λ_n. Note that $[\![s]\!]$ is a clopen set. For all $n \in \mathbb{N}$, let $\mathcal{C}_n$ be the family of cylinder sets on Σ^{Λ_n}, i.e.,

$$\mathcal{C}_n := \{[\![s]\!] : s \in \Sigma^{\Lambda_n}\},$$

and set $\mathcal{C} := \bigcup_n \mathcal{C}_n$. The set $\mathcal{C}$ generates the Borel σ-algebra $\mathfrak{B}(\Omega)$.

We set

$$\Sigma^{\Lambda_*} := \bigcup_{n=0}^{\infty} \Sigma^{\Lambda_n}$$

where $\Sigma^{\Lambda_0} := \{\lambda\}$ and for all $n \in \mathbb{N}, \Sigma^{\Lambda_n} := \{(\omega_g)_{g\in\Lambda_n} : \forall g \in \Lambda_n, \omega_g \in \Sigma\}$, and write $[\![V]\!] := \bigcup_{s\in V}[\![s]\!]$ for all $V \subset \Sigma^{\Lambda_*}$.

Let $\sigma^g : \Omega \to \Omega$ denote the shift by $g \in G$, i.e., $(\sigma^g\omega)_i := \omega_{i+g}$ for all $\omega = (\omega_g)_{g\in G}$, and we write $\sigma := (\sigma^g)_{g\in G}$. Then σ is a continuous action of G on Ω. Hence (Ω, σ) is a t.d.s.. Note that σ is a map from $G \times \Omega$ to Ω, i.e., $\sigma : G \times \Omega \ni (g, \omega) \mapsto \sigma^g(\omega) \in \Omega$.

A nonempty subset $S \subset \Omega$ is called a *subshift* if and only if S is shift-invariant (i.e. $\forall g \in G, \sigma^g(S) = S$) and S is closed. If $S \subset \Omega$ is a subshift, then $(S, \sigma \restriction (G \times S))$ is a t.d.s.. There exists a measure of maximal entropy measure for $\sigma \restriction (G \times S)$ (see [3]).

We call $f : G \to \mathbb{Z}_+$ a *computable function* if there exists a partial recursive prefix function $\phi : \{0,1\}^* \to \{0,1\}^*$ such that for all $(x_1, \cdots, x_d) \in G$,

$$f(x_1, \cdots, x_d) = (I_{\{0,1\}^*\to\mathbb{Z}_+} \circ \phi)\left(\overline{I_{\sharp\to\{0,1\}^*}(x_1)} \cdots \overline{I_{\sharp\to\{0,1\}^*}(x_{d-1})} I_{\sharp\to\{0,1\}^*}(x_d)\right)$$

where

$$\sharp = \begin{cases} \mathbb{Z}, & G = \mathbb{Z}^d, \\ \mathbb{Z}_+, & G = \mathbb{Z}_+^d. \end{cases}$$

We fix an arbitrary bijective computable function $f : G \to \mathbb{Z}_+$ such that for all $n \in \mathbb{N}$,

$$f(\Lambda_n) = \{0, 1, \cdots, |\Lambda_n| - 1\}$$

and define $\mathcal{G} : \Sigma^{\Lambda_*} \to \Sigma^*$ as follows.

$$\mathcal{G}(s) := \begin{cases} s_{f^{-1}(0)} \cdots s_{f^{-1}(|\Lambda_n|-1)}, & s = (s_g)_{g \in \Lambda_n} \in \Sigma^{\Lambda_n} \ (n \in \mathbb{N}), \\ \lambda, & s = \lambda. \end{cases}$$

We define the prefix Kolmogorov complexity of $s \in \Sigma^{\Lambda_*}$ by

$$\mathsf{K}(s) := K(\mathcal{G}(s)).$$

Lemma 2.21 *For all $n \in \mathbb{N}$ and $k \in \mathbb{R}_{\geq 0}$, we define*

$$D_{n,k} := \{s \in \Sigma^{\Lambda_n} : \mathsf{K}(s) < k\}.$$

Then

$$|D_{n,k}| \leq 2^{k+1}.$$

Proof. Note that for all $s \in \Sigma^{\Lambda_n}$ $(n \in \mathbb{N})$, $\mathsf{K}(s) = K_\phi(\mathcal{G}(s)) \neq \infty$. We define $\psi : \Sigma^{\Lambda_n} \to \{0,1\}^*$ such that the following condition holds.

$$\psi(s) = p_s \Longleftrightarrow \phi(p_s) = \mathcal{G}(s) \text{ and } \mathsf{K}(s) = l(p_s).$$

For all $s, t \in \Sigma^{\Lambda_n}$, we have

$$s \neq t \Longrightarrow \phi(p_s) \neq \phi(p_t) \Longrightarrow p_s \neq p_t$$

then ψ is injective. Therefore

$$\begin{aligned} |D_{n,k}| &= |\{s \in \Sigma^{\Lambda_n} : \exists p_s \in \{0,1\}^*, \psi(s) = p_s, l(p_s) < k\}| \\ &\leq |\{p \in \{0,1\}^* : l(p) < k\}| \\ &\leq 1 + 2 + \cdots + 2^{\lfloor k \rfloor} = 2^{\lfloor k \rfloor + 1} - 1 \leq 2^{k+1}. \end{aligned}$$

□

Definition 2.22 (Kolmogorov complexity density) *The upper and lower Kolmogorov complexity density of $\omega \in \Omega$ are defined by*

$$\overline{\mathcal{K}}(\omega) := \limsup_{n \to \infty} \frac{\mathsf{K}(\omega \upharpoonright \Lambda_n)}{|\Lambda_n|}, \quad \underline{\mathcal{K}}(\omega) := \liminf_{n \to \infty} \frac{\mathsf{K}(\omega \upharpoonright \Lambda_n)}{|\Lambda_n|}.$$

If $\overline{\mathcal{K}}(\omega) = \underline{\mathcal{K}}(\omega)$, we simply denote them by $\mathcal{K}(\omega)$, i.e.,

$$\mathcal{K}(\omega) := \lim_{n \to \infty} \frac{\mathsf{K}(\omega \upharpoonright \Lambda_n)}{|\Lambda_n|}.$$

Remark 2.23 *The quantities $\overline{\mathcal{K}}(\omega)$ and $\underline{\mathcal{K}}(\omega)$ are independent of the choice of additively optimal partial recursive prefix function ϕ and $\mathcal{G}$ and uniquely defined.*

3 Relation between KS entropy and Kolmogorov complexity

Let $d \in \mathbb{N}$, $G = \mathbb{Z}^d$ or $G = \mathbb{Z}_+^d$, Σ be a nonempty finite set, and $S \subset \Omega$ $(:= \Sigma^G)$ be a subshift. Other notations are the same as before. We set $\varsigma := \sigma \upharpoonright (G \times S)$. Note that (S, ς) is a t.d.s. We now state the main result.

Theorem 3.1 (Brudno's theorem for $\mathbb{Z}^d$ (or $\mathbb{Z}_+^d$) subshifts) *If $\mu \in EM(S, \varsigma)$, then*

$$\mathcal{K}(\omega) = h_\varsigma(\mu), \quad \mu\text{-}a.e.\omega \in S. \tag{3.1}$$

Remark 3.2 *Brudno's original result is on the case $G = \mathbb{Z}_+$ only [2]. In the case $G = \mathbb{Z}^d$ or $G = \mathbb{Z}_+^d$, Simpson showed that if μ is a measure of maximal entropy, then (3.1) holds [7]. Our theorem is a generalization of them.*

We prove Theorem 3.1 by giving two lemmas.

Lemma 3.3 *If $\mu \in EM(S, \varsigma)$, then*

$$\underline{\mathcal{K}}(\omega) \geq h_\varsigma(\mu), \quad \mu\text{-}a.e.\omega \in S. \tag{3.2}$$

Proof. If $h_\varsigma(\mu) = 0$, then (3.2) is obvious.

Let $h_\varsigma(\mu) > 0$ and fix an arbitrary $k \in \mathbb{N}$ such that $\frac{1}{k} < h_\varsigma(\mu)$. For all $n \in \mathbb{N}$, we set

$$\widetilde{D_{n,k}} := \left\{ s \in \Sigma^{\Lambda_n} : \frac{\mathsf{K}(s)}{|\Lambda_n|} \leq h_\varsigma(\mu) - \frac{1}{k} \right\}.$$

By Lemma 2.21, we have

$$|\widetilde{D_{n,k}}| \leq 2^{|\Lambda_n|(h_\varsigma(\mu) - \frac{1}{k}) + 1}. \tag{3.3}$$

We fix an arbitrary $\epsilon \in \left(0, \frac{1}{k}\right)$ and set

$$T_{n,k,\epsilon} := \left\{ s \in \Sigma^{\Lambda_n} : \mu([\![s]\!] \cap S) < 2^{-|\Lambda_n|(h_\varsigma(\mu) - \frac{1}{k} + \epsilon)} \right\}.$$

By Shannon-McMillan-Breiman theorem (Theorem 2.9), the following holds for μ-a.e. $\omega \in S$

$$\exists N_\omega, \forall n \geq N_\omega, \left| h_\varsigma(\mu) - \frac{-\log_2 \mu([\![\omega \upharpoonright \Lambda_n]\!] \cap S)}{|\Lambda_n|} \right| < \frac{1}{k} - \epsilon.$$

Note that

$$\begin{aligned}\left|h_\varsigma(\mu) - \frac{-\log_2 \mu([\![\omega \restriction \Lambda_n]\!] \cap S)}{|\Lambda_n|}\right| < \frac{1}{k} - \epsilon \quad &\Longrightarrow \quad \frac{-\log_2 \mu([\![\omega \restriction \Lambda_n]\!] \cap S)}{|\Lambda_n|} > h_\varsigma(\mu) - \frac{1}{k} + \epsilon \\ &\Longleftrightarrow \quad 2^{-|\Lambda_n|(h_\varsigma(\mu) - \frac{1}{k} + \epsilon)} > \mu([\![\omega \restriction \Lambda_n]\!] \cap S) \\ &\Longleftrightarrow \quad \omega \restriction \Lambda_n \in T_{n,k,\epsilon} \\ &\Longleftrightarrow \quad \omega \in [\![T_{n,k,\epsilon}]\!] \cap S.\end{aligned}$$

Hence we have for μ-a.e. $\omega \in S$

$$\exists N_\omega, \forall n \geq N_\omega, \omega \in [\![T_{n,k,\epsilon}]\!] \cap S. \tag{3.4}$$

On the other hand, by (3.3) and the definition of $T_{n,k,\epsilon}$, we have

$$\begin{aligned}\mu([\![\widetilde{D_{n,k}}]\!] \cap [\![T_{n,k,\epsilon}]\!] \cap S) &= \mu\left(\bigcup_{s \in \widetilde{D_{n,k}} \cap T_{n,k,\epsilon}} [\![s]\!] \cap S\right) \\ &\leq \sum_{s \in \widetilde{D_{n,k}} \cap T_{n,k,\epsilon}} \mu([\![s]\!] \cap S) \\ &\leq 2^{|\Lambda_n|(h_\varsigma(\mu) - \frac{1}{k}) + 1} \cdot 2^{-|\Lambda_n|(h_\varsigma(\mu) - \frac{1}{k} + \epsilon)} = 2^{-|\Lambda_n|\epsilon + 1}.\end{aligned}$$

Hence

$$\sum_{n=1}^{\infty} \mu([\![\widetilde{D_{n,k}}]\!] \cap [\![T_{n,k,\epsilon}]\!] \cap S) < \infty.$$

Therefore, by the Borel-Cantelli lemma, for μ-a.e. $\omega \in S$,

$$\exists N'_\omega \in \mathbb{N}, \forall n \geq N'_\omega, \omega \notin [\![\widetilde{D_{n,k}}]\!] \cap [\![T_{n,k,\epsilon}]\!] \cap S. \tag{3.5}$$

By (3.4) and (3.5), for μ-a.e. $\omega \in S$, we have

$$\exists N''_\omega \in \mathbb{N}, \forall n \geq N''_\omega, \omega \notin [\![\widetilde{D_{n,k}}]\!]. \tag{3.6}$$

Since $\omega \notin [\![\widetilde{D_{n,k}}]\!]$ means $\frac{\mathsf{K}(\omega \restriction \Lambda_n)}{|\Lambda_n|} > h_\varsigma(\mu) - \frac{1}{k}$, for all $k \in \mathbb{N}$, we have

$$\underline{\mathcal{K}}(\omega) \geq h_\varsigma(\mu) - \frac{1}{k}, \quad \mu\text{-a.e.}\omega \in S. \tag{3.7}$$

Thus (3.2) holds. $\square$

Lemma 3.4 *If $\mu \in EM(S, \varsigma)$, then*

$$\overline{\mathcal{K}}(\omega) \le h_{\varsigma}(\mu), \quad \mu\text{-}a.e.\omega \in S. \tag{3.8}$$

Proof. Fix an arbitrary $m \in \mathbb{N}$ and let L_m be the side length of the hypercube Λ_m, i.e.,

$$L_m := \begin{cases} m & \text{if } G = \mathbb{Z}_+^d, \\ 2m-1 & \text{if } G = \mathbb{Z}^d. \end{cases}$$

For each $n \in \mathbb{N}_{>m}$, let us consider a covering of Λ_n by shifted Λ_m. In particular, there uniquely exists $k \in \mathbb{N}$ such that

$$\bigsqcup_{g \in \Lambda_k} (L_m g + \Lambda_m) \subsetneq \Lambda_n \subset \bigsqcup_{g \in \Lambda_{k+1}} (L_m g + \Lambda_m),$$

where $L_m g + \Lambda_m := \{L_m g + h : h \in \Lambda_m\}$ and $\bigsqcup$ denotes disjoint union. We set $\check{\Lambda}_n := \bigsqcup_{g \in \Lambda_k} (L_m g + \Lambda_m)$ and $M := |\Sigma^{\Lambda_m}|$. Let us consider bijective map $r : \{1, \cdots, |\Sigma^{\Lambda_m}|\} \to \Sigma^{\Lambda_m}$, and let $r_j := r(j)$ $(1 \le j \le M)$. For an arbitrary $\omega \in S$, we define

$$\begin{aligned} \mathsf{f}_{r_j}(\omega) &:= |\{g \in \Lambda_k : \varsigma^{-L_m g}\omega \in [\![r_j]\!] \cap S\}|, \\ \mathsf{f}_r(\omega) &:= (\mathsf{f}_{r_1}(\omega), \cdots, \mathsf{f}_{r_M}(\omega)) \in \mathbb{Z}_+^M. \end{aligned}$$

By definition, $\mathsf{f}_{r_1}(\omega) + \cdots + \mathsf{f}_{r_M}(\omega) = |\Lambda_k|$ holds for all $\omega \in S$. Since for all $s \in \Sigma^{\check{\Lambda}_n}$ and for all $\omega_1, \omega_2 \in [\![s]\!] \cap S$, we have $\mathsf{f}_{r_j}(\omega_1) = \mathsf{f}_{r_j}(\omega_2)$. We set $\mathsf{f}_{r_j}(s) := \mathsf{f}_{r_j}(\omega)$ $(\omega \in [\![s]\!] \cap S)$. We endow $\Sigma^{\check{\Lambda}_n}$ with an equivalence relation as follows:

$$\forall s_1, s_2 \in \Sigma^{\check{\Lambda}_n}, \quad s_1 \sim_{\mathsf{f}_r} s_2 \overset{\text{def}}{\iff} \mathsf{f}_r(s_1) = \mathsf{f}_r(s_2).$$

For $s \in \Sigma^{\check{\Lambda}_n}$, let $[s]_{\mathsf{f}_r} := \{t \in \Sigma^{\check{\Lambda}_n} : s \sim_{\mathsf{f}_r} t\}$ be an equivalence class of s by $\sim_{\mathsf{f}_r}$. Then we have

$$|[s]_{\mathsf{f}_r}| = \frac{|\Lambda_k|!}{\mathsf{f}_{r_1}(s)!\mathsf{f}_{r_2}(s)! \cdots \mathsf{f}_{r_M}(s)!}.$$

By the above mentioned preparations, we take the following procedures:

(1) We fix a bijective map $\mathcal{F}$ from $\Sigma^{\check{\Lambda}_n}/\sim_{\mathsf{f}_r}$ to $V_k^M := \{(x_1, \cdots, x_M) \in \mathbb{Z}_+^M : x_1 + \cdots + x_M = |\Lambda_k|\}$ such that

$$\mathcal{F} : [s]_{\mathsf{f}_r} \mapsto (\mathsf{f}_{r_1}(s), \cdots, \mathsf{f}_{r_M}(s)),$$

and identify $\Sigma^{\check{\Lambda}_n}/\sim_{\mathsf{f}_r}$ with V_k^M.

(2) We fix the arbitrary bijective maps $\mathcal{N}_{[s]_{\mathsf{f}_r}}$ and $\mathcal{R}_{\Lambda_n \setminus \check{\Lambda}_n}$ such that for each $[s]_{\mathsf{f}_r} \in \Sigma^{\check{\Lambda}_n} / \sim_{\mathsf{f}_r}$,

$$\mathcal{N}_{[s]_{\mathsf{f}_r}} : [s]_{\mathsf{f}_r} \to \{1, \cdots, |[s]_{\mathsf{f}_r}|\}$$

and

$$\mathcal{R}_{\Lambda_n \setminus \check{\Lambda}_n} : \Sigma^{\Lambda_n \setminus \check{\Lambda}_n} \to \{1, 2, \cdots, |\Sigma^{\Lambda_n \setminus \check{\Lambda}_n}|\}.$$

Then we can uniquely identify each $t \in \Sigma^{\Lambda_n}$ with $(\mathbf{x}, y, z)$ where $\mathbf{x} \in V_k^M$, $y \in \mathcal{N}_{\mathcal{F}^{-1}(\mathbf{x})}(\mathcal{F}^{-1}(\mathbf{x}))$ and $z \in \{1, 2, \cdots, |\Sigma^{\Lambda_n \setminus \check{\Lambda}_n}|\}$. Hence there exists a partial recursive prefix function $\phi : \{0,1\}^* \to \Sigma^*$ such that

$$\forall \omega \in S, \exists \mathbf{x} = (x_1, \cdots, x_M) \in V_k^M, \exists y \in \mathcal{N}_{\mathcal{F}^{-1}(\mathbf{x})}(\mathcal{F}^{-1}(\mathbf{x})), \exists z \in \{1, 2, \cdots, |\Sigma^{\Lambda_n \setminus \check{\Lambda}_n}|\},$$
$$\phi\left(\overline{I_{\mathbb{Z}_+ \to \{0,1\}^*}(x_1)} \cdots \overline{I_{\mathbb{Z}_+ \to \{0,1\}^*}(x_M)}\ \overline{I_{\mathbb{Z}_+ \to \{0,1\}^*}(y)}\ I_{\mathbb{Z}_+ \to \{0,1\}^*}(z)\right) = \mathcal{G}(\omega \upharpoonright \Lambda_n).$$

Obviously, $x_j = \mathsf{f}_{r_j}(\omega)$ $(j \in \{1, \cdots, M\})$. By the definition of K_ϕ, we have

$$\begin{aligned} K_\phi(\mathcal{G}(\omega \upharpoonright \Lambda_n)) &\le l\left(\overline{I_{\mathbb{Z}_+ \to \{0,1\}^*}(x_1)} \cdots \overline{I_{\mathbb{Z}_+ \to \{0,1\}^*}(x_M)}\ \overline{I_{\mathbb{Z}_+ \to \{0,1\}^*}(z)}\ I_{\mathbb{Z}_+ \to \{0,1\}^*}(y)\right) \\ &\le 2\left(\sum_{j=1}^{M} l(I_{\mathbb{Z}_+ \to \{0,1\}^*}(x_j)) + l(I_{\mathbb{Z}_+ \to \{0,1\}^*}(z))\right) \\ &\quad + M + 1 + l(I_{\mathbb{Z}_+ \to \{0,1\}^*}(y)). \end{aligned} \tag{3.9}$$

The following inequalities can be easily seen:

$$\sum_{j=1}^{M} l(I_{\mathbb{Z}_+ \to \{0,1\}^*}(x_j)) \le \sum_{j=1}^{M} \log_2(x_j + 1) \le M \log_2(|\Lambda_k| + 1),$$
$$l(I_{\mathbb{Z}_+ \to \{0,1\}^*}(z)) \le \log_2(|\Sigma^{\Lambda_n \setminus \check{\Lambda}_n}| + 1) \le (|\Lambda_n| - |\check{\Lambda}_n|) \log_2 |\Sigma| + 1,$$
$$l(I_{\mathbb{Z}_+ \to \{0,1\}^*}(y)) \le \log_2 y + 1 \le \log_2 \frac{|\Lambda_k|!}{x_1! \cdots x_M!} + 1.$$

By these inequalities and (3.9), we have

$$\frac{K_\phi(\mathcal{G}(\omega \upharpoonright \Lambda_n))}{|\Lambda_n|} \le \frac{1}{|\Lambda_n|} \log_2 \frac{|\Lambda_k|!}{x_1! \cdots x_M!} + \frac{2(|\Lambda_n| - |\check{\Lambda}_n|)}{|\Lambda_n|} \log_2 |\Sigma| + o(1), \quad (n \to \infty). \tag{3.10}$$

Let us estimate the right hand side of (3.10). By direct computations using Stirling's formula, we can see that

$$\frac{1}{|\Lambda_n|} \log_2 \frac{|\Lambda_k|!}{x_1! \cdots x_M!} = \frac{|\Lambda_k|}{|\Lambda_n|} \sum_{j=1}^{M} \varphi\left(\frac{x_j}{|\Lambda_k|}\right) + o(1), \quad (n \to \infty). \tag{3.11}$$

Since $x_j = \mathsf{f}_{r_j}(\omega)$ and $\mu \in EM(S, \varsigma)$, for μ-a.e.$\omega \in S$, we have the following.

$$\begin{aligned}
\lim_{k\to\infty} \frac{x_j}{|\Lambda_k|} &= \lim_{k\to\infty} \frac{1}{|\Lambda_k|} |\{g \in \Lambda_k : \varsigma^{-L_m g}\omega \in [\![r_j]\!] \cap S\}| \\
&= \lim_{k\to\infty} \frac{1}{|\Lambda_k|} \sum_{g\in\Lambda_k} 1_{[\![r_j]\!]\cap S}(\varsigma^{-L_m g}\omega) = \mu([\![r_j]\!] \cap S). \quad (3.12)
\end{aligned}$$

The last equality is derived from the Birkhoff's ergodic theorem.

By $\Lambda_{n-L_m} \subset \check{\Lambda}_n \subsetneq \Lambda_n$ and $|\check{\Lambda}_n| = |\Lambda_m| \cdot |\Lambda_k|$, we have $|\Lambda_{n-L_m}| \leq |\Lambda_m| \cdot |\Lambda_k| \leq |\Lambda_n|$. Then

$$\frac{1}{|\Lambda_m|} \cdot \frac{|\Lambda_{n-L_m}|}{|\Lambda_n|} \leq \frac{|\Lambda_k|}{|\Lambda_n|} \leq \frac{1}{|\Lambda_m|}$$

and we have

$$\frac{|\Lambda_{n-L_m}|}{|\Lambda_n|} = \begin{cases} \frac{(n-L_m)^d}{n^d} \to 1 & \text{if } G = \mathbb{Z}_+^d, \\ \frac{\{2(n-L_m)-1\}^d}{(2n-1)^d} \to 1 & \text{if } G = \mathbb{Z}^d, \end{cases} \quad (n \to \infty).$$

Hence the following hold:

$$\lim_{n\to\infty} \frac{|\Lambda_k|}{|\Lambda_n|} = \frac{1}{|\Lambda_m|}, \ \lim_{n\to\infty} \frac{|\check{\Lambda}_n|}{|\Lambda_n|} = 1. \quad (3.13)$$

Obviously, if $n \to \infty$, then $k \to \infty$. By (3.10), (3.11), (3.12), (3.13) and Kolmogorov complexity's definition, for μ-a.e.$\omega \in S$ and for all $m \in \mathbb{N}$, we have

$$\begin{aligned}
\overline{\mathcal{K}}(\omega) &\leq \limsup_{n\to\infty} \frac{K_\phi(\mathcal{G}(\omega \upharpoonright \Lambda_n))}{|\Lambda_n|} \\
&\leq \limsup_{n\to\infty} \left\{ \frac{|\Lambda_k|}{|\Lambda_n|} \sum_{j=1}^{M} \varphi\left(\frac{x_j}{|\Lambda_k|}\right) + \frac{2(|\Lambda_n| - |\check{\Lambda}_n|)}{|\Lambda_n|} \log_2 |\Sigma| \right\} \\
&= \frac{1}{|\Lambda_m|} \sum_{j=1}^{M} \varphi(\mu([\![r_j]\!] \cap S)).
\end{aligned}$$

Hence we have the following inequality:

$$\overline{\mathcal{K}}(\omega) \leq \liminf_{m\to\infty} \frac{1}{|\Lambda_m|} \sum_{j=1}^{M} \varphi(\mu([\![r_j]\!] \cap S)), \quad \text{for } \mu\text{-a.e. } \omega \in S. \quad (3.14)$$

Note that $\alpha := \{[\![\omega \upharpoonright \Lambda_1]\!] \cap S\}_{\omega \in S}$ is a μ-generator and

$$\alpha^{\Lambda_m} = \bigvee_{g \in \Lambda_m} \varsigma^{-g} \alpha = \{[\![r_j]\!] \cap S\}_{j=1}^{M}.$$

Therefore, by Kolmogorov-Sinai theorem, we have the following equation.

$$\lim_{m \to \infty} \frac{1}{|\Lambda_m|} \sum_{j=1}^{M} \varphi(\mu([\![r_j]\!] \cap S)) = \lim_{m \to \infty} \frac{1}{|\Lambda_m|} H_\mu(\alpha^{\Lambda_m}) = h_\varsigma(\mu). \tag{3.15}$$

(3.14) and (3.15) complete the proof. $\square$

Theorem 3.1 follows from Lemma 3.3 and Lemma 3.4.

Example 3.5 (d-dimensional Bernoulli shifts) *Let (Ω, σ) be the d-dimensional shift space as before. We fix a probability vector $q = (q_i : i \in \Sigma)$ on Σ and denote the corresponding Bernoulli measure on $\mathfrak{B}(\Omega)$ by $\mu := q^{\times G}$. Then, by Kolmogorov-Sinai theorem, we can show that $h_\sigma(\mu) = \sum_{i \in \Sigma} \varphi(q_i)$. By Theorem 3.1, we have for μ-a.e. $\omega \in \Omega$*

$$\mathcal{K}(\omega) = \sum_{i \in \Sigma} \varphi(q_i).$$

4 Representation of the pressure by using Kolmogorov complexity density

Let us consider some applications of Theorem 3.1. Notations are the same as in Section 3.

Theorem 4.1 *If $\mu \in M(S, \varsigma)$, then we have*

$$h_\varsigma(\mu) = \mu(\mathcal{K}) = \lim_{n \to \infty} \frac{1}{|\Lambda_n|} \sum_{s \in \Lambda_n} \mathsf{K}(s) \mu([\![s]\!] \cap S). \tag{4.1}$$

Proof. Let $\mu = \int_{EM(S,\varsigma)} \nu d\rho(\nu)$ be the ergodic decomposition. By Theorem 2.17, Jacobs's theorem (Theorem 2.18) and the main theorem (Theorem 3.1), we have

$$\int_S \mathcal{K}(\omega) d\mu(\omega) = \int_{EM(S,\varsigma)} \left\{ \int_S \mathcal{K}(\omega) d\nu(\omega) \right\} d\rho(\nu) = \int_{EM(S,\varsigma)} h_\varsigma(\nu) d\rho(\nu) = h_\varsigma(\mu).$$

On the other hand, by Lebesgue's convergence theorem, we have

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{|\Lambda_n|}\sum_{s\in\Lambda_n}\mathsf{K}(s)\mu([\![s]\!]\cap S) &= \lim_{n\to\infty}\int_S\frac{\mathsf{K}(\omega\restriction\Lambda_n)}{|\Lambda_n|}d\mu(\omega)\\ &= \int_S\lim_{n\to\infty}\frac{\mathsf{K}(\omega\restriction\Lambda_n)}{|\Lambda_n|}d\mu(\omega)=\int_S\mathcal{K}(\omega)d\mu(\omega).\end{aligned}$$

□

Remark 4.2 *In the case $G=\mathbb{Z}_+$, Theorem 4.1 can be found in [1].*

Theorem 4.1 immediately leads one to the following theorem.

Theorem 4.3 (Variational principle) *Let $\psi\in USC(S)$, $\inf\psi>-\infty$. Then the pressure of ψ is given by*

$$p(\psi)=\sup_{\mu\in M(S,\varsigma)}\mu(\mathcal{K}+\psi).$$

In particular, the topological entropy is $\sup_{\mu\in M(S,\varsigma)}\mu(\mathcal{K})$. If $\mu\in M(S,\varsigma)$ is an equilibrium state for ψ, then we have

$$p(\psi)=\mu(\mathcal{K}+\psi).$$

Theorem 4.3 shows that, in an equilibrium state, the pressure means the expectation value of the sum of Kolmogorov complexity density and local energy. For example, this theorem is directly applicable to the d-dimensional Ising model.

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